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A Geometric Theory of Surface Area

Part II: Triangulable Parametric Surfaces

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A.

In [1] we presented a geometric theory of the area of non-parametric surfaces. We showed that this theory is equivalent to the analytic theory of LEBESGUE for such surfaces. In the present paper we extend this geometric theory to triangulable parametric surfaces. Since every surface may be thought of as a parametric surface and as one may expect that the transition from triangulable surfaces to non-triangulable ones may be made by a limit process, the present theory may be considered as an introduction to a general geometric theory of surface area.

We shall use the term *parametric surface* to mean the locus in $\mathscr{E}^3$ of a system of simultaneous equations $x=f(u,v)$, $y=g(u,v)$, $z=h(u,v)$, these functions being defined and continuous on E, a closed set in the uv plane consisting of the interior and the boundary of a closed simple polygon, this set being a minimal preimage of the given surface. We shall make use of triangular polyhedra inscribed in the given surface and such that every face of the polyhedra has an angle which lies between a prescribed angle φ and $\pi-\varphi$, $0<\varphi<\pi$. We refer to such polyhedra as admissible polyhedra. However, since we limit our discussions to such polyhedra, we shall omit the term „admissible", except when we wish to make special emphasis of it.

These polyhedra have a finite number of faces. By the area of such a polyhedron, we mean the sum of the areas of its faces. We shall index these faces and thus write $T_1, T_2, \ldots, T_n$ for the faces of a polyhedron.

Definitions:

1. A triangle $T \subset \mathscr{E}^3$ is said to be *inscribed in a surface* S if its three vertices all lie in S. A polyhedron Π is said to be inscribed in S, if all of its faces (closed triangles) are inscribed in S.

2. Let Π be inscribed in S. Let T be a face of Π. By $P_{xy}(T)$, we shall mean the orthogonal projection of T on the xy coordinate plane. We define $P_{xy}(T)$ and $P_{yz}(T)$ similarly. By $S_{xy}(T)$ we shall mean a maximal connected subset of S whose orthogonal projection on the xy plane is a subset of $P_{xy}(T)$ and which, moreover, has the property that no two distinct points of $S_{xy}(T)$ have identical projection points on the xy plane. It is seen that for each inscribed triangle T, there may be arbitrarily associated a single $S_{xy}(T)$, or a finite or even a countably infinite set of $S_{xy}(T)$. We shall denote the union of this set by $S_{xy}{}^*(T)$. We define $S_{xz}(T)$, $S_{xz}{}^*(T)$, $S_{yz}(T)$ and $S_{yz}{}^*(T)$ similarly. We refer to $S_{xy}{}^*(T) \cup S_{xz}{}^*(T) \cup S_{yz}{}^*(T)$ as a portion of S which is subtended by the inscribed triangle T. By $P_{xy}(\Pi)$ we shall mean $P_{xy}(T_1) \cup P_{xy}(T_2) \cup \cdots \cup P_{xy}(T_n)$, when the faces of Π are $T_1, T_2, \cdots, T_n$. We define $P_{xz}(\Pi)$ and $P_{yz}(\Pi)$ similarly. By $S_{xy}{}^*(\Pi)$ we mean $S_{xy}{}^*(T_1) \cup \cup S_{xy}{}^*(T_2) \cup \cdots \cup S_{xy}{}^*(T_n)$. We define $S_{xz}{}^*(\Pi)$ and $S_{yz}{}^*(\Pi)$ similarly.

3. If

a) $S_{xy}{}^*(\Pi) \cup S_{xz}{}^*(\Pi) \cup S_{yz}{}^*(\Pi) = S$ and
b) the set $\{S_{xy}{}^*(T_1), S_{xy}{}^*(T_2), \cdots, S_{xy}{}^*(T_n)\}$ is disjoint except for boundary points; the set $\{S_{xz}{}^*(T_1), S_{xz}{}^*(T_2), \cdots, S_{xz}{}^*(T_n)\}$ is disjoint except for boundary points; the set $\{S_{yz}{}^*(T_1), S_{yz}{}^*(T_2), \cdots, S_{yz}{}^*(T_n)\}$ is disjoint except for boundary points, then we say that Π is *inscribed on* the surface S. By the decomposition norm of a polyhedron Π inscribed on S we shall mean the greatest of the diameters of its faces.

A surface S is said to be triangulable at a given point $Q \in S$ if for every ball $S(Q, \varepsilon)$ there exists an admissible triangle $T \subset S(Q, \varepsilon)$ which is inscribed in S. A surface S is said to be triangulable if it is triangulable at each of its points. In this paper we shall confine ourselves to such surfaces.

4. Let T be a face of a polyhedron Π inscribed on S. By $B_{xy}(T)$ we mean the area of $P_{xy}(T)$. We define $B_{xz}(T)$ and $B_{yz}(T)$ similarly. Let $B_{xy}(\Pi) = B_{xy}(T_1) + B_{xy}(T_2) + \cdots + B_{xy}(T_n)$. We define $B_{xz}(\Pi)$ and $B_{yz}(\Pi)$ similarly. If, for all polyhedra Π that

may be inscribed on S, the sets {all $B_{xy}(\Pi)$}, {all $B_{xz}(\Pi)$} and {all $B_{yz}(\Pi)$} are each bounded, then we say that S is *tame*. In this case, we designate the LUB of {all $B_{xy}(\Pi)$} by $B_{xy}(S)$. We define $B_{xz}(S)$ and $B_{yz}(S)$ similarly. We designate the sum $B_{xy}(S)+ + B_{xz}(S)+B_{yz}(S)$ by B and call it the *base area* of S.

It is easy to see [2] that if S is not tame, then its Lebesgue area is infinite. In the sequel we shall confine ourselves to tame triangulable surfaces.

5. Let T be a face of a polyhedron Π inscribed on S. By $D(T)$, the *deviation* on T, we mean the LUB of the set of the acute angles between the normal to T and the normals to the admissible triangles which may be inscribed in $S_{xy}^*(T) \cup S_{xz}^*(T) \cup S_{yz}^*(T)$.

6. Let Π be a polyhedron inscribed on S. By $D(\Pi)$, the *deviation norm* of Π, we mean the largest of the deviations on its faces.

7. Let $Q \in S$. By $D(Q)$, the deviation at Q, we mean the GLB of the set $\{D(T)\}$, T any admissible triangle which may be inscribed in the intersection of S and a ball $S(Q,\varepsilon)$, $\varepsilon > 0$. Since we are dealing exclusively with triangulable surfaces, $D(Q)$ is defined at every point of S.

8. S is said to be *piecewise flat* if for every $\varepsilon > 0$, there exists an admissible polyhedron Π inscribed on S such that the deviation norm of Π is less than ε.

It is clear that if S is piecewise flat then there exists a sequence $(\Pi_1, \Pi_2, \cdots)$ of admissible polyhedra inscribed on S such that the corresponding sequence $(N_1, N_2, \cdots)$ of the deviation norms converges to zero. We shall call the sequence $(\Pi_1, \Pi_2, \cdots)$ *a regular sequence of inscribed polyhedra.*

9. S is said to be *quasi-piecewise flat* (*qpf*) if for every $\alpha > 0$ and every $\beta > 0$, there exists a polyhedron Π inscribed on S such that

(a) for each of some of the faces of Π (the so-called α-regular faces of Π), the deviation is less than α and

(b) the sum of the areas of the faces of Π on which the deviations are not less than α, is less than β.

It is clear that if S is *qpf* then there exists a sequence $(\Pi_1, \Pi_2, \cdots)$ of polyhedra inscribed on S such that the corresponding sequences $(\alpha_1, \alpha_2, \cdots)$ and $(\beta_1, \beta_2, \cdots)$ both converge to zero. We shall also call such a sequence $(\Pi_1, \Pi_2, \cdots)$ a regular sequence of inscribed polyhedra.

10. Let α, β, γ be the direction angles of a vector in $\mathscr{E}^3$. Since $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$, it follows that one of the coordinate axes makes with the given vector an acute angle which is no larger than $\alpha^* = \arccos 1/\sqrt{3}$. Thus, for every plane in $\mathscr{E}^3$, one of the coordinate planes makes with this plane a dihedral angle which is less than or equal to α^*.

Let S be *qpf* and $(\Pi_1, \Pi_2, \cdots)$ be a regular sequence of polyhedra inscribed on S. Consider, in succession, the faces T_{n1}, T_{n2}, $\cdots$, T_{nm_n} of Π_n. For each T_{ni} there exists a coordinate plane which makes with T_{ni} a dihedral angle $\theta \leqslant \alpha^*$. As one runs through the sequence $(T_{n1}, T_{n2}, \cdots, T_{nm_n})$, select the faces which make with the xy plane angles $\theta \leqslant \alpha^*$. We denote the set of these faces F_{nxy}. We define the sets F_{nxz} and F_{nyz} similarly. One can make further selections so that these three sets are disjoint.

Consider the sequences

$$\Pi_n, \Pi_{n+1}, \cdots$$
$$F_{nxy}, F_{(n+1)xy}, \cdots$$
$$F_{nyz}, F_{(n+1)yz}, \cdots$$
$$F_{nxz}, F_{(n+1)xz}, \cdots$$

The sequence $(F_{nxy}, F_{(n+1)xy}, \cdots)$ is a sequence of polyhedra inscribed in S, for which the angle between the z-axis and the normal to each face of each polyhedron is less than or equal to α^*. $(F_{nxy}, F_{(n+1)xy}, \cdots)$ is a strongly regular sequence of inscribed polyhedra in the sense of [1]. Similarly, $(F_{nxz}, F_{(n+1)xz}, \cdots)$ and $(F_{nyz}, F_{(n+1)yz}, \cdots)$ are strongly regular sequences of polyhedra inscribed in S.

11. Given a polyhedron Π inscribed on S, by a *refinement* of Π we mean a polyhedron Π^* also inscribed on S such that every vertex of Π is also a vertex of Π^*.

Given two polyhedra Π_1 and Π_2 both inscribed on S, one may construct a *common refinement* Π^* of Π_1 and Π_2 in the following manner:

Project on the xy plane the vertices of the set $F_{nxy}, n = 1, 2, \cdots, n_{m_1}$ and the vertices $F_{nxy}, n = 1, 2, \cdots, n_{m_2}$ of Π_2. These points on the xy plane with the addition, if necessary, of a set of well chosen points (see [1]) on the xy plane determine an admissible polyhedron inscribed on the portion of S which is subtended by the union of the F_{nxy} of Π_1 and the F_{nxy} of Π_2. Project on the xz plane the vertices of the F_{nxz} of Π_1 and the vertices of the F_{nxz} of Π_2 and proceed in like manner. Project on the yz plane the vertices of the F_{nyz} of Π_1 and also the vertices of F_{nyz} of Π_2 and proceed in like manner. This procedure leads to the construction of a third admissible polyhedron Π^* inscribed on S which is a common refinement of Π_1 and Π_2.

B. We now state

Theorem 1.

Let S be such that $D(Q)=0$ for every $Q \in S$. Then there exists a sequence $(\Pi_1, \Pi_2, \cdots)$ of polyhedra inscribed on S such that the corresponding sequence $(N_1, N_2, \ldots)$ of deviation norms converges to zero. For all such sequences of polyhedra inscribed on S, the corresponding sequence $(A_1, A_2, \cdots)$ of the polyhedral areas converges to a unique real number. This is independent of the admissibility number φ.
Proof:

Let $\varepsilon > 0$ be given. For each $Q \in S$ there exists a ball $S(Q, \delta)$ such that, if T_1 and T_2 are any two triangles inscribed in $S \cap S(Q, \delta)$, then the dihedral angle between T_1 and T_2 is less than $\varepsilon/2$. Associate to the point Q the ball $S(Q, (\delta/2))$. Letting Q run over S gives us a covering Γ of S. Since S is compact, there exists a finite subcovering Γ^* of S. Using this covering Γ^* we obtain a polyhedron Π inscribed on S such that the deviation norm of Π is less than ε. By considering a sequence $(\varepsilon_1, \varepsilon_2, \cdots)$ which converges to zero, we obtain a sequence $(\Pi_1, \Pi_2, \ldots)$ of polyhedra inscribed on S such that the corresponding sequence of the deviation norms converges to zero.

To show that the corresponding sequence of the polyhedral areas converges to a unique real number, we make use of two lemmas.

Lemma 1.

Let $(\Pi_1, \Pi_2, \cdots)$ be a sequence of polyhedra inscribed on S such that the corresponding sequence $(N_1, N_2, \ldots)$ of the deviation norms converges to zero. There exists a positive integer N such that, if $n > N$, then $\alpha_n < \frac{1}{2}((\pi/2) - \alpha^*)$. Let $n > N$. Let T be any face of the F_{nxy} of Π_n. Let k be any refinement of T. Let θ denote the acute angle between the z-axis and the normal to T. Then there exists a positive constant $M_{(xy)N}$ such that $|\sec\theta - \sec\theta_i| < M_{(xy)N} |\theta - \theta_i|$, where θ_i is the acute angle between the z-axis and the normal to any face T_i of k.

Proof:

$\theta \leqslant \alpha^*$ and $\alpha^* - \frac{1}{2}((\pi/2)-\alpha^*) \leqslant \theta_i < \alpha^* + \frac{1}{2}((\pi/2)-\alpha^*)$. $\sec\theta$ is uniformly Lipschitzian on the closed interval $[\alpha^* - \frac{1}{2}((\pi/2)-\alpha^*), \alpha + \frac{1}{2}((\pi/2)-\alpha^*)]$.

Lemma 2.

Let T be any α_n-regular face of the F_{xy} of Π_n, $n > N$, as in Lemma 1. Let K be any refinement of T. Let the faces of K be $T_1, T_2, \cdots, T_{nm}$ and let their respective areas be $A_1, A_2, \cdots, A_{nm}$. Let a_{xy} denote the area of $P_{xy}(T)$ and $a_{(xy)i}$ the area of $P_{xy}(T_i)$. Let A denote the area of T and let $A^* = A_1 + A_2 + \cdots + A_{nm}$. Then $|A - A^*| < a_{xy} M_{(xy)N} \alpha_n$.

Proof:

$$A^* = a_{(xy)1} \sec\theta_1 + \ldots + a_{(xy)nm} \sec\theta_{nm}$$

$$A = a_{(xy)1} \sec\theta + \cdots + a_{(xy)nm} \sec\theta$$

$$\begin{aligned} |A - A^*| &\leqslant a_{(xy)1} |\sec\theta - \sec\theta_1| + \cdots + a_{(xy)nm} |\sec\theta - \sec\theta_{nm}| \\ &< a_{(xy)1} M_{(xy)N} \alpha_n + \cdots + a_{(xy)nm} M_{(xy)N} \alpha_n \\ &< a_{xy} M_{(xy)N} \alpha_n . \end{aligned}$$

If, in the above lemmas, we had considered a face T in the F_{xz} of Π, we would have arrived at the same results with $M_{(xz)N}$ and a_{xz} instead of $M_{(xy)N}$ and a_{xy}, respectively. Had we considered a face T in the F_{yz} of Π, we would have arrived at similar results. Let M_N denote the greatest of $M_{(xy)N}, M_{(xz)N}$ and $M_{(yz)N}$.

Corollary 1.

Let $n > N$ as in Lemma 1. Let $\Pi_n{}^*$ be any refinement of Π_n. Let A_n denote the sum of the areas of the faces of Π_n and $A_n{}^*$ the sum of the areas of the faces of $\Pi_n{}^*$ which are subtended by the faces of Π_n. Then $|A_n - A_n{}^*| < M_n B \alpha_n$.

We now proceed to the proof of Theorem 1.

Let $\varepsilon > 0$ be given. There exists a positive integer N such if $n > N$, then $\alpha_n < (\varepsilon/4 M_N B)$. Let $n_1 > N$ and $n_2 > N$. Let $\Pi_N{}^*$ be a common refinement of Π_{n_1} and Π_{n_2}. Let A_{n_1}, A_{n_2} and $A_N{}^*$ be the areas of Π_{n_1}, Π_{n_2} and $\Pi_N{}^*$, respectively. Then $|A_{n_1} - A_N{}^*| < < (\varepsilon/4 M_N B) \cdot M_N B = (\varepsilon/4)$. Similarly $|A_{n_2} - A_N{}^*| < (\varepsilon/4)$. Hence $|A_{n_1} - A_{n_2}| < (\varepsilon/2)$. Thus the sequence $(A_1, A_2, \cdots)$ converges to a unique real number. It is easy to see that the sequential limit is independent of the particular admissibility number.

Corollary 2.

Let S be piecewise flat. Let $(\Pi_1, \Pi_2, \cdots)$ be a regular sequence of polyhedra inscribed on S. Then the corresponding sequence $(A_1, A_2, \ldots)$ of the polyhedral areas converges to a unique real number.

Definition.

Let S be *qpf*. S is said to be *sqpf* (*strongly quasipiecewise flat*) if there exists a sequence of polyhedra $(\Pi_1, \Pi_2, \cdots)$ inscribed on S such that, for each n, in addition to conditions 9(a) and 9(b) above, the dihedral angle between any of the α_n-irregular faces of Π_n and any refinement of such a face has an upper bound which is less than $(\pi/2)$. We shall refer to such a sequence of polyhedra as a *strongly regular sequence.*

Theorem 2.

Let S be sqpf. Let $(\Pi_1, \Pi_2, \ldots)$ be a strongly regular sequence of polyhedra inscribed on S. Then the corresponding sequence $(A_1, A_2, \cdots)$ of the polyhedral areas converges to a unique real number.

Proof:

The secants of the dihedral angles referred to in the above definition are bounded. Let m denote the upper bound of these secants. The proof is now essentially the same as that of Theorem 1 of [1].

C.

We now describe the analytic significance of $D(Q) = 0$, $Q \in S$.

Let $S = F(E)$, E satisfying the conditions stated at the beginning of Section A.

Set $D(Q) = 0$. Let $P \in E$ such that $Q = F(P)$. Let P be (x_0, y_0). Consider the line $u = u_0$ on the uv plane. The image of this line under the transformation F is a curve in $\mathscr{E}^3$. By arguments similar to those used in the proofs of Theorems 3 and 4 of [1], one easily shows that $\frac{\partial x}{\partial u}, \frac{\partial x}{\partial v}, \frac{\partial y}{\partial u}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial u}, \frac{\partial z}{\partial v}$ all exist at P, and with respect to the domains of these partial derivatives, are continuous at P.

Theorem 3.

Let S be such that, for all $Q \in S$, $D(Q) = 0$. Then for every sequence of polyhedra inscribed on S, such that the corresponding sequences of *decomposition norms and deviation norms both converge to zero, the limit of the sequence* $(A_1, A_2, \ldots)$ *of the polyhedral areas is given by the Riemann double integral* $\int_E \sqrt{J_1^2 + J_2^2 + J_3^2}\, d(u,v)$ *where*

$$J_1 = \frac{\partial(y,z)}{\partial(u,v)}, \quad J_2 = \frac{\partial(z,x)}{\partial(u,v)}, \quad J_3 = \frac{\partial(x,y)}{\partial(u,v)}.$$

Proof:

At every point $P \in E$, $\frac{\partial x}{\partial u}, \frac{\partial x}{\partial v}, \frac{\partial y}{\partial u}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial u}$, and $\frac{\partial z}{\partial v}$ exist and are continuous. Since E is bounded and closed, each of these partial derivatives is uniformly continuous on E.

Let P: (u_0, v_0), $Q = (f(u_0, v_0), g(u_0, v_0), h(u_0, v_0))$. Let $S = F(E)$. The transformation F carries the lines $u = u_0$ and $v = v_0$ to curves on S. Call these C_u and C_v, respectively. These curves have tangent lines at $F(P)$. Call them L_u and L_v, respectively. At $F(P)$, L_u and L_v have direction numbers $(f_u(u_0,v_0), g_u(u_0,v_0), h_u(u_0,v_0))$ and $(f_v(u_0,v_0), g_v(u_0,v_0), h_v(u_0,v_0))$. The cosine of the angle between L_u and L_v is given by

$$\cos\theta = \frac{f_u f_v + g_u g_v + h_u h_v}{\sqrt{f_u^2 + g_u^2 + h_u^2}\,\sqrt{f_v^2 + g_v^2 + h_v^2}}.$$

We consider decompositions of E by right triangles with legs parallel to the uv coordinate axes and with arbitrarily small diameters.

Let the vertices of a typical such right triangle be (u_0, v_0),

$(u_0, v_0 + \Delta v)$, $(u_0 + \Delta u, v_0)$. Let $F((u_0, v_0)) = (x_0, y_0, z_0)$;

$F(u_0, v_0 + \Delta v) = (x_0 + \Delta_v x, y_0 + \Delta_v y, z_0 + \Delta_v z)$;

$F(u_0 + \Delta u, v_0) = (x_0 + \Delta_u x, y_0 + \Delta_u y, z_0 + \Delta_u z)$.

The area of the triangle determined by these three points on the surface is given by

$$\sqrt{\overline{\Delta_v x}^2 + \overline{\Delta_v y}^2 + \overline{\Delta_v z}^2}\,\sqrt{\overline{\Delta_u x}^2 + \overline{\Delta_u y}^2 + \overline{\Delta_u z}^2}\,\sin\beta,$$

where β is the appropriate angle.

Now, let $\varepsilon > 0$ be given. There exists $\delta > 0$ such that if $|\Delta u| < \delta$ and $|\Delta v| < \delta$, then

$$\Big| \sqrt{\overline{\Delta_v x}^2 + \overline{\Delta_v y}^2 + \overline{\Delta_v z}^2} \sqrt{\overline{\Delta_u x}^2 + \overline{\Delta_u y}^2 + \overline{\Delta_u z}^2} \sin\beta -$$

$$\sqrt{f_v^2 + g_v^2 + h_v^2} \sqrt{f_u^2 + g_u^2 + h_u^2}\, |\Delta u| |\Delta v| \sin\theta \Big| < (\varepsilon/B).$$

Consider now a sequence of decompositions $(D_1, D_2, \cdots)$ of E with corresponding sequence of decompositions norms converging to zero and the following corresponding sequences:

$D_1, D_2, , ,$

$N_1, N_2, , ,$

$\Sigma_1, \Sigma_2, , ,$

$\Sigma_1', \Sigma_2', , , ,$ where the Σ are the areas of triangles inscribed on S and the Σ' are sums of the type

$$\sum \sqrt{f_v^2 + g_v^2 + h_v^2} \sqrt{f_u^2 + g_u^2 + h_u^2} \sin\theta.$$ Now

$\sqrt{f_v^2 + g_v^2 + h_v^2} \sqrt{f_u^2 + g_u^2 + h_u^2} \sin\theta$ may be written in the form

$$\sqrt{(f_u^2 + g_u^2 + h_u^2)(f_v^2 + g_v^2 + h_v^2) - (f_u f_v + g_u g_v + h_u h_v)^2} =$$

$$= \sqrt{J_1^2 + J_2^2 + J_3^2}.$$

It follows that the sequence $(\Sigma_1', \Sigma_2', \cdots)$ converges to $\iint_E \sqrt{J_1^2 + J_2^2 + J_3^2}\, du dv$. Hence, the sequence $(\Sigma_1, \Sigma_2, \cdots)$ also converges to the same double integral.

We now consider the case where there exists $P \in E$ such that $D(F(P)) > 0$.

Theorem 4.

Let $G = \{$all $P \in E$ such that $D(F(P)) > 0\}$. If G is of Lebesgue measure zero, then there exists a sequence $(\Pi_1, \Pi_2, \cdots)$ of polyhedra inscribed on S such that the corresponding sequence $(A_1, A_2, \cdots)$ of the polyhedral areas converges to the Lebesgue area of S, whether this be finite or infinite.

Proof:

The set G is closed and bounded and hence compact. For every $\varepsilon > 0$ there exists a closed set $E_\varepsilon \subset E$, consisting of the interior and the boundary of a simple polygon, such that, for all points $P \in E_\varepsilon$, $D(F(P))=0$ and *area of* E — *area of* $E_\varepsilon < \varepsilon$. By Theorems 2 and 3 there exists a sequence of polyhedra $(\Pi_1, \Pi_2, \cdots)$ inscribed on E_ε such that the corresponding sequence $(A_1, A_2, \cdots)$ of the polyhedral areas converges to the integral $\int\limits_{E_\varepsilon} \sqrt{J_1^2+J_2^2+J_3^2}\, d\,(u,v)$. This is precisely the Lebesgue area L_ε of $S_\varepsilon = F(E_\varepsilon)$.

Consider now a sequence $(\varepsilon_1, \varepsilon_2, \cdots)$ of positive numbers converging to zero.

Let $(\Pi_{11}, \Pi_{12}, \cdots)$ be a sequence of polyhedra inscribed on $S_{\varepsilon_1} = F(E_{\varepsilon_1})$. On $\overline{E - E_{\varepsilon_1}}$ there exists a finite triangulation A_1 of area less than ε_2 which contains all the points $P \in E$ for which $D(F(P))>0$. Let $E_{\varepsilon_2} = E_{\varepsilon_1} \cup \overline{(E - E_{\varepsilon_1} - A_1)}$. There exists a sequence $(A_{21}, A_{22}, \cdots)$ of polyhedra inscribed on $S_{\varepsilon_2} = F(E_{\varepsilon_2})$ such that the corresponding sequence $(A_{21}, A_{22}, \cdots)$ of the polyhedral areas converges to the integral $\int\limits_{E_{\varepsilon_2}} \sqrt{J_1^2+J_2^2+J_3^2}\, d(u,v)$.

Continuing this process indefinitely gives us a sequence of sequences

$\varepsilon_1 : \Pi_{11}, \Pi_{12}, \cdots$

$$A_{11}, A_{12}, \cdots \text{ converges to } \int\limits_{E_{\varepsilon_1}} \sqrt{J_1^2+J_2^2+J_3^2}\, d(u,v);$$

$$\varepsilon_2 : \Pi_{21}, \Pi_{22}, \cdots \text{ converges to } \int\limits_{E_{\varepsilon_2}} \sqrt{J_1^2+J_2^2+J_3^2}\, d(u,v); \ldots$$

Consider now the sequence $\left(\int\limits_{E_{\varepsilon_1}}, \int\limits_{E_{\varepsilon_2}}, \cdots\right)$. Here $\int\limits_{E_{\varepsilon_1}} \leqslant \int\limits_{E_{\varepsilon_2}} \cdots$. If this sequence is unbounded, then, by the additivity of Lebesgue area, the Lebesgue area of S is infinite.

Now suppose that the sequence $\left(\int\limits_{E_{\varepsilon_1}}, \int\limits_{E_{\varepsilon_2}}, \cdots\right)$ is bounded. Then this sequence converges to a real number. Since the Lebesgue integral $\int\limits_{E} \sqrt{J_1^2+J_2^2+J_3^2}\, d(u,v)$ exists, the sequence $\left(\int\limits_{E_{\varepsilon_1}}, \int\limits_{E_{\varepsilon_2}}, \cdots\right)$ converges to $\int\limits_{E} \sqrt{J_1^2+J_2^2+J_3^2}\, d(u,v)$.

We now wish to set up a sequence of polyhedra $(\Pi_1^*, \Pi_2^*, \ldots)$ inscribed on S such that the corresponding sequence $(A_1^*, A_2^*, \cdots)$ converges to the Lebesgue area of S.

Π_1^* is built from Π_{11} by merely adjoining a polyhedron on $\overline{E - E_{\varepsilon_1}} \cdot \Pi_2^*$ is built from Π_{22}^* by merely adjoining a polyhedron on $\overline{E - E_{\varepsilon_2}}$, etc. Since the sequence $\left(\int\limits_{E_{\varepsilon_1}}, \int\limits_{E_{\varepsilon_2}}, \cdots\right)$ converges to $\int\limits_E \sqrt{J_1^2 + J_2^2 + J_3^2}\, d(u,v)$, the sequence $(A_1^*, A_2^*, \cdots)$ also converges to $\int\limits_E \sqrt{J_1^2 + J_2^2 + J_3^2}\, d(u,v)$. It follows (from e of [2]) that this limit is the Lebesgue area of S.

The identical procedure followed in the case where the sequence $\left(\int\limits_{E_{\varepsilon_1}}, \int\limits_{E_{\varepsilon_2}}, \cdots\right)$ is unbounded yields the limit ∞ which is the Lebesgue area of S.

We now consider the case where the set G is of positive outer Lebesgue measure. For this we have the following theorem.

Theorem 5.

If the set G of points P of E for which $D(F(P)) > 0$ is of positive outer Lebesgue measure, then the Lebesgue area of S is infinite.

Proof:

Consider the projection $P_{xy}(S)$, $P_{xz}(S)$, and $P_{yz}(S)$ of S on the xy, xz, yz coordinate planes, respectively. Since the set H of points Q of S for which $D(Q) > 0$ is of positive outer Lebesgue measure, it follows that on one of the three projection sets $P_{xy}(S)$, $P_{xz}(S)$ and $P_{yz}(S)$ there is a set of points R of positive outer Lebesgue measure which are the projection of points Q of S for which $D(Q) > 0$. It now follows from Theorem 9 of [1] that this portion of S is of infinite Lebesgue area. It follows that the Lebesgue area of S is infinite.

In this paper we have confined ourselves to triangulable parametric surfaces. Some surfaces are not triangulable, e. g., degenerate surfaces such as the loci in $\mathscr{E}^3$ of functions $x = f(u,v)$, $y = f(u,v)$, $s = f(u,v)$. We shall give a treatment of these surfaces in a succeeding article.

References

[1] Toralballa, L. V.: A Geometric Theory of Surface Area. Part I: Non-parametric surfaces. Monatsh. Math. **74**, 462—476 (1970).

[2] Toralballa, L. V.: Piecewise Flatness and Surface Area. Annales Polonici Mathematici **21**, 223—230 (1969).

[3] Toralballa, L. V.: Directional Deviation Norms and Surface Area. L'Enseignement Mathématique **13**, 111—118 (1967).

[4] Radó, T.: Length and Area. American Mathematical Society Colloquium Publications. 1948.

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